

Multiple reference states and complete spectrum of the \mathbb{Z}_n Belavin model with open boundaries

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Abstract

The multiple reference state structure of the \mathbb{Z}_n Belavin model with non-diagonal boundary terms is discovered. It is found that there exist n reference states, each of them yields a set of eigenvalues and Bethe Ansatz equations of the transfer matrix. These n sets of eigenvalues together constitute the complete spectrum of the model. In the quasi-classic limit, they give the complete spectrum of the corresponding Gaudin model.

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1 Introduction

Our understanding of quantum phase transitions and critical phenomena has been greatly enhanced by the study of exactly solvable models (integrable models) [1]. Such exact results provide valuable insights into the key theoretical development of universality classes in areas ranging from modern condensed physics [2] to string and super-symmetric Yang-Mills theories [3]. Among solvable models, elliptic ones stand out as a particularly important class due to the fact that most other ones can be obtained from them by some trigonometric or rational limits. In this paper, we focus on the elliptic \mathbb{Z}_n Belavin model [4] with integrable boundary conditions, with the celebrated XYZ spin chain as the special case of $n = 2$.

Two-dimensional integrable models have traditionally been solved by imposing periodic boundary conditions. For such bulk systems, the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \quad (1.1)$$

leads to families of commuting row-to-row transfer matrix which may be diagonalized by the quantum inverse scattering method (QISM) (or algebraic Bethe Ansatz) [5].

Not all boundary conditions are compatible with integrability in the bulk. The bulk integrability is only preserved when one imposes certain boundary conditions. In [6], Sklyanin developed the boundary QISM, which may be used to describe integrable systems on a finite interval with independent boundary conditions at each end. This boundary QISM uses a new algebraic structure, the reflection equation (RE) algebra. The solutions to the RE and its dual are called boundary K-matrices which in turn give rise to boundary conditions compatible with the integrability of the bulk model [6, 7, 8].

The boundary QISM has been successfully applied to diagonalize the double-row transfer matrices of various integrable models with non-trivial boundary conditions mostly corresponding to the diagonal K-matrices. The problem of diagonalizing the double-row transfer matrix for most general *non-diagonal* K-matrices by the algebraic Bethe Ansatz has been long-standing due to the difficulty of finding suitable reference states (or pseudo-vacuum states). Recently, much progress has been made for the open XXZ spin chain. Bethe Ansatz solutions for non-diagonal boundary terms where the boundary parameters obey some constraints have been proposed by various approaches¹ [9, 10, 11, 12, 13, 14, 15, 16]. It has been

¹Solutions with arbitrary boundary parameters were recently proposed by functional Bethe Ansatz [17] and q-Onsager algebra [18]. However, it seems highly non-trivial to rederive these results in the framework of algebraic Bethe Ansatz.

found that in order to obtain the complete spectrum of the model two sets of Bethe Ansatz equations and consequently two sets of eigenvalues are *needed* [19, 15], in contrast with the diagonal boundary case [6]. This suggests that in the framework of algebraic Bethe Ansatz there should exist two reference states corresponding to the two sets of Bethe Ansatz equations and eigenvalues. Such multiple reference state structure was confirmed in our recent work [20].

However, for models related to higher rank algebras [21, 22, 23, 24, 25, 26] only one reference state, and consequently only one set of eigenvalues and Bethe Ansatz equations of their transfer matrices, have been constructed so far. It is natural to expect that there exist *extra* reference states, giving rise to extra sets of eigenvalues and associated Bethe Ansatz equations for such models. In this paper we investigate the multiple reference state structure for an elliptic model related to A_{n-1} algebra - the \mathbb{Z}_n Belavin model with general non-diagonal boundary terms. It is found that there actually *exist* n reference states for such a model. Each of these reference states yields a set of eigenvalues and corresponding Bethe Ansatz equations of the transfer matrix. The n sets of eigenvalues together constitute the complete spectrum of the model. In the quasi-classic limit, they give the corresponding spectrum of the associated Gaudin model [27].

The paper is organised as follows. In section 2, we briefly review the \mathbb{Z}_n Belavin model with integrable open boundary conditions, which also serves as an introduction to our notion and basic ingredients. In section 3, we introduce the intertwiner vectors and the corresponding face-vertex correspondence relations which will play key roles in transforming the model in “vertex picture” to the one in the “face picture”. In section 4, after finding the n reference states, we use the algebraic Bethe Ansatz to obtain the corresponding n sets of eigenvalues and the associated n sets of Bethe Ansatz equations of the transfer matrix of the model. In section 5, we take the quasi-classic limit to extract the spectrum of the associated Gaudin operators. Section 6 is for conclusion. The Appendix provides the definitions of some elementary functions appeared in section 4 and 5.

2 \mathbb{Z}_n Belavin model with open boundaries

Let us fix a positive integer $n \geq 2$, a complex number τ such that $\text{Im}(\tau) > 0$ and a generic complex number w . Introduce the following elliptic functions

$$\theta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (u, \tau) = \sum_{m=-\infty}^{\infty} \exp \{ \sqrt{-1} \pi [(m+a)^2 \tau + 2(m+a)(u+b)] \}, \quad (2.1)$$

$$\theta^{(j)}(u) = \theta \left[\begin{smallmatrix} \frac{1}{2} - \frac{j}{n} \\ \frac{1}{2} \end{smallmatrix} \right] (u, n\tau), \quad \sigma(u) = \theta \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix} \right] (u, \tau), \quad (2.2)$$

$$\zeta(u) = \frac{\partial}{\partial u} \{ \ln \sigma(u) \}. \quad (2.3)$$

Among them the σ -function² satisfies the following identity:

$$\begin{aligned} & \sigma(u+x)\sigma(u-x)\sigma(v+y)\sigma(v-y) - \sigma(u+y)\sigma(u-y)\sigma(v+x)\sigma(v-x) \\ &= \sigma(u+v)\sigma(u-v)\sigma(x+y)\sigma(x-y). \end{aligned}$$

Let g, h , be $n \times n$ matrices with the elements

$$h_{ij} = \delta_{i+1j}, \quad g_{ij} = \omega^i \delta_{ij}, \quad \text{with} \quad \omega = e^{\frac{2\pi\sqrt{-1}}{n}}, \quad i, j \in \mathbb{Z}_n.$$

For any $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in \mathbb{Z}_n$, one introduces an $n \times n$ matrix I_α by

$$I_\alpha = I_{(\alpha_1, \alpha_2)} = g^{\alpha_2} h^{\alpha_1},$$

and an elliptic function $\sigma_\alpha(u)$ by

$$\sigma_\alpha(u) = \theta \left[\begin{smallmatrix} \frac{1}{2} + \frac{\alpha_1}{n} \\ \frac{1}{2} + \frac{\alpha_2}{n} \end{smallmatrix} \right] (u, \tau), \quad \text{and} \quad \sigma_{(0,0)}(u) = \sigma(u).$$

The \mathbb{Z}_n Belavin R-matrix is [4, 28]

$$R^B(u) = \frac{\sigma(w)}{\sigma(u+w)} \sum_{\alpha \in \mathbb{Z}_n^2} \frac{\sigma_\alpha(u + \frac{w}{n})}{n\sigma_\alpha(\frac{w}{n})} I_\alpha \otimes I_\alpha^{-1}. \quad (2.4)$$

The R-matrix satisfies the QYBE (1.1) and the properties [28],

$$\text{Unitarity :} \quad R_{12}^B(u) R_{21}^B(-u) = \text{id}, \quad (2.5)$$

$$\text{Crossing-unitarity :} \quad (R^B)_{21}^{t_2}(-u - nw) (R^B)_{12}^{t_2}(u) = \frac{\sigma(u)\sigma(u+nw)}{\sigma(u+w)\sigma(u+nw-w)} \text{id}, \quad (2.6)$$

$$\text{Quasi-classical property :} \quad R_{12}^B(u)|_{w \rightarrow 0} = \text{id}. \quad (2.7)$$

²Our σ -function is the ϑ -function $\vartheta_1(u)$ [29]. It has the following relation with the *Weierstrassian* σ -function denoted by $\sigma_w(u)$: $\sigma_w(u) \propto e^{\eta_1 u^2} \sigma(u)$, $\eta_1 = \pi^2(\frac{1}{6} - 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}})$ and $q = e^{\sqrt{-1}\tau}$. Consequently, our ζ -function (2.3) is different from the *Weierstrassian* ζ -function by an additional term $-2\eta_1 u$.

Here $R_{21}^B(u) = P_{12}R_{12}^B(u)P_{12}$ with P_{12} being the usual permutation operator and t_i denotes the transposition in the i -th space. Here and below we adopt the standard notation: for any matrix $A \in \text{End}(\mathbb{C}^n)$, A_j is an operator embedded in the tensor space $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots$, which acts as A on the j -th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the i -th and j -th ones. The quasi-classical properties (2.7) of the R-matrix enables one to introduce the associated classical \mathbb{Z}_n r-matrix $r(u)$ [30]

$$\begin{aligned} R^B(u) &= \text{id} + w r(u) + O(w^2), \quad \text{when } w \longrightarrow 0, \\ r(u) &= \frac{1-n}{n} \zeta(u) + \sum_{\alpha \in \mathbb{Z}_n^2 - (0,0)} \frac{\sigma'(0)\sigma_\alpha(u)}{n\sigma(u)\sigma_\alpha(0)} I_\alpha \otimes I_\alpha^{-1}, \quad \sigma'(0) = \frac{\partial}{\partial u} \sigma(u)|_{u=0}. \end{aligned} \quad (2.8)$$

In the above equation, the elliptic ζ -function is defined in (2.3).

One introduces the “row-to-row” monodromy matrix $T(u)$ [5], which is an $n \times n$ matrix with elements being operators acting on $(\mathbb{C}^n)^{\otimes N}$

$$T(u) = R_{01}^B(u + z_1) R_{02}^B(u + z_2) \cdots R_{0N}^B(u + z_N). \quad (2.9)$$

Here $\{z_i | i = 1, \dots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters. With the help of the QYBE (1.1), one can show that $T(u)$ satisfies the so-called “RLL” relation

$$R_{12}^B(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}^B(u - v). \quad (2.10)$$

An integrable open chain can be constructed as follows [6]. Let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former satisfies the RE

$$\begin{aligned} R_{12}^B(u_1 - u_2) K_1^-(u_1) R_{21}^B(u_1 + u_2) K_2^-(u_2) \\ = K_2^-(u_2) R_{12}^B(u_1 + u_2) K_1^-(u_1) R_{21}^B(u_1 - u_2), \end{aligned} \quad (2.11)$$

and the latter satisfies the dual RE

$$\begin{aligned} R_{12}^B(u_2 - u_1) K_1^+(u_1) R_{21}^B(-u_1 - u_2 - nw) K_2^+(u_2) \\ = K_2^+(u_2) R_{12}^B(-u_1 - u_2 - nw) K_1^+(u_1) R_{21}^B(u_2 - u_1). \end{aligned} \quad (2.12)$$

For the models with open boundaries, instead of the standard “row-to-row” monodromy matrix $T(u)$ (2.9), one needs the “double-row” monodromy matrix $\mathbb{T}(u)$

$$\mathbb{T}(u) = T(u) K^-(u) T^{-1}(-u). \quad (2.13)$$

Using (2.10) and (2.11), one can prove that $\mathbb{T}(u)$ satisfies

$$R_{12}^B(u_1 - u_2)\mathbb{T}_1(u_1)R_{21}^B(u_1 + u_2)\mathbb{T}_2(u_2) = \mathbb{T}_2(u_2)R_{12}^B(u_1 + u_2)\mathbb{T}_1(u_1)R_{21}^B(u_1 - u_2). \quad (2.14)$$

Then the *double-row transfer matrix* of the inhomogeneous \mathbb{Z}_n Belavin model with open boundary is given by

$$\tau(u) = \text{tr}(K^+(u)\mathbb{T}(u)). \quad (2.15)$$

The commutativity of the transfer matrices

$$[\tau(u), \tau(v)] = 0, \quad (2.16)$$

follows as a consequence of (1.1)-(2.6) and (2.11)-(2.12). This ensures the integrability of the inhomogeneous \mathbb{Z}_n Belavin model with open boundary.

In this paper, we consider a *non-diagonal* K-matrix $K^-(u)$ which is a solution to the RE (2.11) associated with the \mathbb{Z}_n Belavin R-matrix [31]

$$K^-(u)_t^s = \sum_{i=1}^n \frac{\sigma(\lambda_i + \xi - u)}{\sigma(\lambda_i + \xi + u)} \phi_{\lambda, \lambda - w\hat{i}}^{(s)}(u) \bar{\phi}_{\lambda, \lambda - w\hat{i}}^{(t)}(-u). \quad (2.17)$$

The corresponding *dual* K-matrix $K^+(u)$ which is a solution to the dual RE (2.12) has been obtained in [32]. With a particular choice of the free boundary parameters with respect to $K^-(u)$, we introduce the corresponding dual K-matrix $K^+(u)$

$$K^+(u)_t^s = \sum_{i=1}^n \left\{ \prod_{k \neq i} \frac{\sigma((\lambda_i - \lambda_k) - w)}{\sigma(\lambda_i - \lambda_k)} \right\} \frac{\sigma(\lambda_i + \bar{\xi} + u + \frac{nw}{2})}{\sigma(\lambda_i + \bar{\xi} - u - \frac{nw}{2})} \times \phi_{\lambda, \lambda - w\hat{i}}^{(s)}(-u) \tilde{\phi}_{\lambda, \lambda - w\hat{i}}^{(t)}(u). \quad (2.18)$$

In (2.17) and (2.18), $\phi, \bar{\phi}, \tilde{\phi}$ are intertwiners which will be specified in section 4. We consider the generic $\{\lambda_i\}$ such that $\lambda_i \neq \lambda_j$ (*modulo* $\mathbb{Z} + \tau\mathbb{Z}$) for $i \neq j$. This condition is necessary for the non-singularity of $K^\pm(u)$. It is convenient to introduce a vector $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$ associated with the boundary parameters $\{\lambda_i\}$, where $\{\epsilon_i, i = 1, \dots, n\}$ is the orthonormal basis of the vector space \mathbb{C}^n such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. Moreover, in the following we always assume that the suffix index of the parameter λ_i takes value in \mathbb{Z}_n cyclic group, namely,

$$\lambda_{i \pm n} = \lambda_i, \quad i = 1, \dots, n. \quad (2.19)$$

3 $A_{n-1}^{(1)}$ SOS R-matrix and face-vertex correspondence

The A_{n-1} simple roots $\{\alpha_i\}$ can be expressed in terms of the orthonormal basis $\{\epsilon_i\}$ as:

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i = 1, \dots, n-1,$$

and the fundamental weights $\{\Lambda_i \mid i = 1, \dots, n-1\}$ satisfying $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$ are given by

$$\Lambda_i = \sum_{k=1}^i \epsilon_k - \frac{i}{n} \sum_{k=1}^n \epsilon_k.$$

Set

$$\hat{i} = \epsilon_i - \bar{\epsilon}, \quad \bar{\epsilon} = \frac{1}{n} \sum_{k=1}^n \epsilon_k, \quad i = 1, \dots, n, \quad \text{then} \quad \sum_{i=1}^n \hat{i} = 0. \quad (3.1)$$

For each dominant weight $\Lambda = \sum_{i=1}^{n-1} a_i \Lambda_i$, $a_i \in \mathbb{Z}^+$ (the set of non-negative integers), there exists an irreducible highest weight finite-dimensional representation V_Λ of A_{n-1} with the highest vector $|\Lambda\rangle$. For example the fundamental vector representation is V_{Λ_1} .

Let \mathfrak{h} be the Cartan subalgebra of A_{n-1} and \mathfrak{h}^* be its dual. A finite dimensional diagonalizable \mathfrak{h} -module is a complex finite dimensional vector space W with a weight decomposition $W = \bigoplus_{\mu \in \mathfrak{h}^*} W[\mu]$, so that \mathfrak{h} acts on $W[\mu]$ by $xv = \mu(x)v$, for any $x \in \mathfrak{h}$, $v \in W[\mu]$. For example, the fundamental vector representation $V_{\Lambda_1} = \mathbb{C}^n$, the non-zero weight spaces $W[\hat{i}] = \mathbb{C}\epsilon_i$, $i = 1, \dots, n$.

For a generic $m \in \mathbb{C}^n$, define

$$m_i = \langle m, \epsilon_i \rangle, \quad m_{ij} = m_i - m_j = \langle m, \epsilon_i - \epsilon_j \rangle, \quad i, j = 1, \dots, n. \quad (3.2)$$

Let $R(u, m) \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the R-matrix of the $A_{n-1}^{(1)}$ SOS model [33],

$$R(u, m) = \sum_{i=1}^n R_{ii}^{ii}(u, m) E_{ii} \otimes E_{ii} + \sum_{i \neq j} \{ R_{ij}^{ij}(u, m) E_{ii} \otimes E_{jj} + R_{ij}^{ji}(u, m) E_{ji} \otimes E_{ij} \}, \quad (3.3)$$

where E_{ij} is the matrix with elements $(E_{ij})_k^l = \delta_{jk} \delta_{il}$. The coefficient functions are

$$R_{ii}^{ii}(u, m) = 1, \quad R_{ij}^{ij}(u, m) = \frac{\sigma(u)\sigma(m_{ij} - w)}{\sigma(u + w)\sigma(m_{ij})}, \quad i \neq j, \quad (3.4)$$

$$R_{ij}^{ji}(u, m) = \frac{\sigma(w)\sigma(u + m_{ij})}{\sigma(u + w)\sigma(m_{ij})}, \quad i \neq j, \quad (3.5)$$

and m_{ij} are defined in (3.2). The R-matrix satisfies the dynamical (modified) QYBE

$$\begin{aligned} R_{12}(u_1 - u_2, m - wh^{(3)}) R_{13}(u_1 - u_3, m) R_{23}(u_2 - u_3, m - wh^{(1)}) \\ = R_{23}(u_2 - u_3, m) R_{13}(u_1 - u_3, m - wh^{(2)}) R_{12}(u_1 - u_2, m), \end{aligned} \quad (3.6)$$

and the quasi-classical property

$$R(u, m)|_{w \rightarrow 0} = \text{id}. \quad (3.7)$$

We adopt the notation: $R_{12}(u, m - wh^{(3)})$ acts on a tensor $v_1 \otimes v_2 \otimes v_3$ as $R(u, m - w\mu) \otimes \text{id}$ if $v_3 \in W[\mu]$. Moreover, the R-matrix satisfies the unitarity and the modified crossing-unitarity relation.

Let us introduce n intertwiner vectors which are n -component column vectors $\phi_{m, m-w\hat{j}}(u)$ labelled by \hat{j} ($j = 1, \dots, n$). The k -th element of $\phi_{m, m-w\hat{j}}(u)$ is given by

$$\phi_{m, m-w\hat{j}}^{(k)}(u) = \theta^{(k)}(u + nm_j). \quad (3.8)$$

We remark that the n intertwiner vectors $\phi_{m, m-w\hat{j}}(u)$ are linearly independent for a generic $m \in \mathbb{C}^n$.

Using the intertwining vector, one derives the following face-vertex correspondence relation [33]

$$\begin{aligned} R_{12}^B(u_1 - u_2) \phi_{m, m-w\hat{i}}(u_1) \otimes \phi_{m-w\hat{i}, m-w(\hat{i}+\hat{j})}(u_2) \\ = \sum_{k,l} R(u_1 - u_2, m)_{ij}^{kl} \phi_{m-w\hat{l}, m-w(\hat{l}+\hat{k})}(u_1) \otimes \phi_{m, m-w\hat{l}}(u_2). \end{aligned} \quad (3.9)$$

Then the QYBE (1.1) of the \mathbb{Z}_n Belavin's R-matrix $R^B(u)$ is equivalent to the dynamical Yang-Baxter equation (3.6) of the $A_{n-1}^{(1)}$ SOS R-matrix $R(u, m)$. For a generic m , we may introduce other types of intertwiners $\bar{\phi}, \tilde{\phi}$ satisfying the conditions,

$$\sum_{k=1}^n \bar{\phi}_{m, m-w\hat{\mu}}^{(k)}(u) \phi_{m, m-w\hat{\nu}}^{(k)}(u) = \delta_{\mu\nu}, \quad (3.10)$$

$$\sum_{k=1}^n \tilde{\phi}_{m+w\hat{\mu}, m}^{(k)}(u) \phi_{m+w\hat{\nu}, m}^{(k)}(u) = \delta_{\mu\nu}, \quad (3.11)$$

from which one can derive the relations,

$$\sum_{\mu=1}^n \bar{\phi}_{m, m-w\hat{\mu}}^{(i)}(u) \phi_{m, m-w\hat{\mu}}^{(j)}(u) = \delta_{ij}, \quad (3.12)$$

$$\sum_{\mu=1}^n \tilde{\phi}_{m+w\hat{\mu}, m}^{(i)}(u) \phi_{m+w\hat{\mu}, m}^{(j)}(u) = \delta_{ij}. \quad (3.13)$$

With the help of (3.10)-(3.13), we obtain the following relations from the face-vertex correspondence relation (3.9):

$$\begin{aligned} & \left(\tilde{\phi}_{m+w\hat{k},m}(u_1) \otimes \text{id} \right) R_{12}^B(u_1 - u_2) \left(\text{id} \otimes \phi_{m+w\hat{j},m}(u_2) \right) \\ &= \sum_{i,l} R(u_1 - u_2, m)_{ij}^{kl} \tilde{\phi}_{m+w(i+\hat{j}),m+w\hat{j}}(u_1) \otimes \phi_{m+w(\hat{k}+\hat{l}),m+w\hat{k}}(u_2), \end{aligned} \quad (3.14)$$

$$\begin{aligned} & \left(\tilde{\phi}_{m+w\hat{k},m}(u_1) \otimes \tilde{\phi}_{m+w(\hat{k}+\hat{l}),m+w\hat{k}}(u_2) \right) R_{12}^B(u_1 - u_2) \\ &= \sum_{i,j} R(u_1 - u_2, m)_{ij}^{kl} \tilde{\phi}_{m+w(i+\hat{j}),m+w\hat{j}}(u_1) \otimes \tilde{\phi}_{m+w\hat{j},m}(u_2), \end{aligned} \quad (3.15)$$

$$\begin{aligned} & \left(\text{id} \otimes \bar{\phi}_{m,m-w\hat{l}}(u_2) \right) R_{12}^B(u_1 - u_2) \left(\phi_{m,m-w\hat{i}}(u_1) \otimes \text{id} \right) \\ &= \sum_{k,j} R(u_1 - u_2, m)_{ij}^{kl} \phi_{m-w\hat{l},m-w(\hat{k}+\hat{l})}(u_1) \otimes \bar{\phi}_{m-w\hat{i},m-w(i+\hat{j})}(u_2), \end{aligned} \quad (3.16)$$

$$\begin{aligned} & \left(\bar{\phi}_{m-w\hat{l},m-w(\hat{k}+\hat{l})}(u_1) \otimes \bar{\phi}_{m,m-w\hat{l}}(u_2) \right) R_{12}^B(u_1 - u_2) \\ &= \sum_{i,j} R(u_1 - u_2, m)_{ij}^{kl} \bar{\phi}_{m,m-w\hat{i}}(u_1) \otimes \bar{\phi}_{m-w\hat{i},m-w(i+\hat{j})}(u_2). \end{aligned} \quad (3.17)$$

The face-vertex correspondence relations (3.9) and (3.14)-(3.17) will play an important role in *translating* formulas in the “vertex form” into those in the “face form”.

Corresponding to the vertex type K-matrices (2.17) and (2.18), one introduces the following face type K-matrices \mathcal{K} and $\tilde{\mathcal{K}}$ [32]

$$\mathcal{K}(\lambda|u)_i^j = \sum_{s,t} \tilde{\phi}_{\lambda-w(i-\hat{j}), \lambda-w\hat{i}}^{(s)}(u) K(u)_t^s \phi_{\lambda, \lambda-w\hat{i}}^{(t)}(-u), \quad (3.18)$$

$$\tilde{\mathcal{K}}(\lambda|u)_i^j = \sum_{s,t} \bar{\phi}_{\lambda, \lambda-w\hat{j}}^{(s)}(-u) \tilde{K}(u)_t^s \phi_{\lambda-w(\hat{j}-\hat{i}), \lambda-w\hat{j}}^{(t)}(u). \quad (3.19)$$

Through straightforward calculations, we find the face type K-matrices *simultaneously* have *diagonal* forms³

$$\mathcal{K}(\lambda|u)_i^j = \delta_i^j k(\lambda|u; \xi)_i, \quad \tilde{\mathcal{K}}(\lambda|u)_i^j = \delta_i^j \tilde{k}(\lambda|u)_i, \quad (3.20)$$

where functions $k(\lambda|u; \xi)_i, \tilde{k}(\lambda|u)_i$ are given by

$$k(\lambda|u; \xi)_i = \frac{\sigma(\lambda_i + \xi - u)}{\sigma(\lambda_i + \xi + u)}, \quad (3.21)$$

$$\tilde{k}(\lambda|u)_i = \left\{ \prod_{k \neq i, k=1}^n \frac{\sigma(\lambda_{ik} - w)}{\sigma(\lambda_{ik})} \right\} \frac{\sigma(\lambda_i + \bar{\xi} + u + \frac{nw}{2})}{\sigma(\lambda_i + \bar{\xi} - u - \frac{nw}{2})}. \quad (3.22)$$

³As will be seen below (see (4.27)), the spectral parameter u and the boundary parameter ξ of the reduced double-row monodromy matrices constructed from $\mathcal{K}(\lambda|u)$ will be shifted in each step of the nested Bethe Ansatz procedure [21]. Therefore, it is convenient to specify the dependence on the boundary parameter ξ of $\mathcal{K}(\lambda|u)$ in addition to the spectral parameter u .

Moreover, one can check that the matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ satisfy the SOS type reflection equation and its dual, respectively [32]. Although the K-matrices $K^\pm(u)$ given by (2.17) and (2.18) are generally non-diagonal (in the vertex form), after the face-vertex transformations (3.18) and (3.19), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ become *simultaneously* diagonal. This fact enables the authors to apply the generalized algebraic Bethe Ansatz method developed in [21] for SOS type integrable models to diagonalize the transfer matrix $\tau(u)$ (2.15).

4 Algebraic Bethe Ansatz

By means of (3.12), (3.13), (3.19) and (3.20), the transfer matrix $\tau(u)$ (2.15) can be recast into the following face type form:

$$\begin{aligned}
\tau(u) &= \text{tr}(K^+(u)\mathbb{T}(u)) \\
&= \sum_{\mu,\nu} \text{tr} \left(K^+(u) \phi_{\lambda-w(\hat{\mu}-\hat{\nu}),\lambda-w\hat{\mu}}(u) \tilde{\phi}_{\lambda-w(\hat{\mu}-\hat{\nu}),\lambda-w\hat{\mu}}(u) \mathbb{T}(u) \phi_{\lambda,\lambda-w\hat{\mu}}(-u) \bar{\phi}_{\lambda,\lambda-w\hat{\mu}}(-u) \right) \\
&= \sum_{\mu,\nu} \bar{\phi}_{\lambda,\lambda-w\hat{\mu}}(-u) K^+(u) \phi_{\lambda-w(\hat{\mu}-\hat{\nu}),\lambda-w\hat{\mu}}(u) \tilde{\phi}_{\lambda-w(\hat{\mu}-\hat{\nu}),\lambda-w\hat{\mu}}(u) \mathbb{T}(u) \phi_{\lambda,\lambda-w\hat{\mu}}(-u) \\
&= \sum_{\mu,\nu} \tilde{\mathcal{K}}(\lambda|u)_\nu^\mu \mathcal{T}(\lambda|u)_\mu^\nu = \sum_\mu \tilde{k}(\lambda|u)_\mu \mathcal{T}(\lambda|u)_\mu^\mu.
\end{aligned} \tag{4.1}$$

Here we have introduced the face-type double-row monodromy matrix $\mathcal{T}(\lambda|u)$,

$$\begin{aligned}
\mathcal{T}(\lambda|u)_\mu^\nu &= \tilde{\phi}_{\lambda-w(\hat{\mu}-\hat{\nu}),\lambda-w\hat{\mu}}(u) \mathbb{T}(u) \phi_{\lambda,\lambda-w\hat{\mu}}(-u) \\
&\equiv \sum_{i,j} \tilde{\phi}_{\lambda-w(\hat{\mu}-\hat{\nu}),\lambda-w\hat{\mu}}^{(j)}(u) \mathbb{T}(u)_i^j \phi_{\lambda,\lambda-w\hat{\mu}}^{(i)}(-u).
\end{aligned} \tag{4.2}$$

This face-type double-row monodromy matrix can be expressed in terms of the face type R-matrix $R(u, \lambda)$ (3.3) and the K-matrix $\mathcal{K}(\lambda|u)$ (3.18) [21]. Moreover from (2.14), (3.9) and (3.13) one may derive the following exchange relations among $\mathcal{T}(m|u)_\mu^\nu$:

$$\begin{aligned}
&\sum_{i_1, i_2} \sum_{j_1, j_2} R(u_1 - u_2, m)_{i_1 j_1}^{i_0 j_0} \mathcal{T}(m + w(\hat{j}_1 + \hat{i}_2)|u_1)_{i_2}^{i_1} \\
&\quad \times R(u_1 + u_2, m)_{j_2 i_3}^{j_1 i_2} \mathcal{T}(m + w(\hat{j}_3 + \hat{i}_3)|u_2)_{j_3}^{j_2} \\
&= \sum_{i_1, i_2} \sum_{j_1, j_2} \mathcal{T}(m + w(\hat{j}_1 + \hat{i}_0)|u_2)_{j_1}^{j_0} R(u_1 + u_2, m)_{i_1 j_2}^{i_0 j_1} \\
&\quad \times \mathcal{T}(m + w(\hat{j}_2 + \hat{i}_2)|u_1)_{i_2}^{i_1} R(u_1 - u_2, m)_{j_3 i_3}^{j_2 i_2}.
\end{aligned} \tag{4.3}$$

Following [21] and motivated by our recent work [20] for the open XXZ chain, we introduce n sets of operators $\{\mathcal{A}^{(s)}, \mathcal{B}^{(s)}, \mathcal{C}^{(s)}, \mathcal{D}^{(s)}\}$, labelled by $s = 1, \dots, n$, as follows:

$$\mathcal{A}^{(s)}(m|u) = \mathcal{T}(m|u)_s^s, \quad \mathcal{B}_i^{(s)}(m|u) = \frac{\mathcal{T}(m|u)_i^s}{\sigma(m_{is})}, \quad \mathcal{C}_i^{(s)}(m|u) = \frac{\mathcal{T}(m|u)_s^i}{\sigma(m_{si})}, \quad i \neq s, \quad (4.4)$$

$$\mathcal{D}_i^{(s)j}(m|u) = \frac{\sigma(m_{js} - \delta_{ij}w)}{\sigma(m_{is})} \left\{ \mathcal{T}(m|u)_i^j - \delta_i^j R(2u, m + w\hat{s})_{sj}^j \mathcal{A}^{(s)}(m|u) \right\}, \quad i, j \neq s. \quad (4.5)$$

Some remarks are in order. Among the n sets of operators, the first set (corresponding to $s = 1$) is the very one which was used in [21, 23] to construct the algebraic Bethe Ansatz. Such algebraic Bethe Ansatz based on the first set of operators only gives rise to the first set of eigenvalues and associated Bethe Ansatz equations. In order to find the complete sets of eigenvalues, we find that the whole n sets of operators $\{\mathcal{A}^{(s)}, \mathcal{B}^{(s)}, \mathcal{C}^{(s)}, \mathcal{D}^{(s)} | s = 1, \dots, n\}$ are needed.

After tedious calculations analogous to those in [21], we have found the commutation relations among $\mathcal{A}^{(s)}(m|u)$, $\mathcal{D}^{(s)}(m|u)$ and $\mathcal{B}^{(s)}(m|u)$ from (4.3). Here we give those which are relevant for our purpose

$$\begin{aligned} & \mathcal{A}^{(s)}(m|u) \mathcal{B}_i^{(s)}(m+w(\hat{i}-\hat{s})|v) \\ &= \frac{\sigma(u+v)\sigma(u-v-w)}{\sigma(u+v+w)\sigma(u-v)} \mathcal{B}_i^{(s)}(m+w(\hat{i}-\hat{s})|v) \mathcal{A}^{(s)}(m+w(\hat{i}-\hat{s})|u) \\ & \quad - \frac{\sigma(w)\sigma(2v)}{\sigma(u-v)\sigma(2v+w)} \frac{\sigma(u-v-m_{si}+w)}{\sigma(m_{si}-w)} \mathcal{B}_i^{(s)}(m+w(\hat{i}-\hat{s})|u) \mathcal{A}^{(s)}(m+w(\hat{i}-\hat{s})|v) \\ & \quad - \frac{\sigma(w)}{\sigma(u+v+w)} \sum_{\alpha \neq s} \frac{\sigma(u+v+m_{\alpha s}+2w)}{\sigma(m_{\alpha s}+w)} \mathcal{B}_\alpha^{(s)}(m+w(\hat{\alpha}-\hat{s})|u) \mathcal{D}_i^{(s)\alpha}(m+w(\hat{i}-\hat{s})|v), \\ & \quad i \neq s, \end{aligned} \quad (4.6)$$

$$\begin{aligned} & \mathcal{D}_i^{(s)k}(m|u) \mathcal{B}_j^{(s)}(m+w(\hat{j}-\hat{s})|v) \\ &= \frac{\sigma(u-v+w)\sigma(u+v+2w)}{\sigma(u-v)\sigma(u+v+w)} \\ & \quad \times \left\{ \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2 \neq s} R(u+v+w, m-w\hat{i})_{\alpha_2 \beta_1}^k \beta_2 R(u-v, m+w\hat{j})_{j i}^{\beta_1 \alpha_1} \right. \\ & \quad \times \mathcal{B}_{\beta_2}^{(s)}(m+w(\hat{k}+\hat{\beta}_2-\hat{i}-\hat{s})|v) \mathcal{D}_{\alpha_1}^{(s)\alpha_2}(m+w(\hat{j}-\hat{s})|u) \Big\} \\ & \quad - \frac{\sigma(w)\sigma(2u+2w)}{\sigma(u-v)\sigma(2u+w)} \left\{ \sum_{\alpha, \beta \neq s} \frac{\sigma(u-v+m_{s\alpha}-w)}{\sigma(m_{s\alpha}-w)} R(2u+w, m-w\hat{i})_{\alpha i}^{k\beta} \right. \\ & \quad \times \mathcal{B}_\beta^{(s)}(m+w(\hat{k}+\hat{\beta}-\hat{i}-\hat{s})|u) \mathcal{D}_j^{(s)\alpha}(m+w(\hat{j}-\hat{s})|v) \Big\} \end{aligned}$$

$$\begin{aligned}
& + \frac{\sigma(w)\sigma(2v)\sigma(2u+2w)}{\sigma(u+v+w)\sigma(2v+w)\sigma(2u+w)} \\
& \times \left\{ \sum_{\alpha \neq s} \frac{\sigma(u+v+m_{sj})}{\sigma(m_{sj}-w)} R(2u+w, m-w\hat{i})_{ji}^{k\alpha} \right. \\
& \quad \left. \times \mathcal{B}_\alpha^{(s)}(m+w(\hat{k}+\hat{\alpha}-\hat{i}-\hat{s})|u) \mathcal{A}^{(s)}(m+w(\hat{j}-\hat{s})|v) \right\}, \\
& \quad i, j, k \neq s,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
& \mathcal{B}_i^{(s)}(m+w(\hat{i}-\hat{s})|u) \mathcal{B}_j^{(s)}(m+w(\hat{i}+\hat{j}-2\hat{s})|v) \\
& = \sum_{\alpha, \beta \neq s} R(u-v, m-2w\hat{s})_{ji}^{\beta\alpha} \mathcal{B}_\beta^{(s)}(m+w(\hat{\beta}-\hat{s})|v) \mathcal{B}_\alpha^{(s)}(m+w(\hat{\alpha}+\hat{\beta}-2\hat{s})|u), \\
& \quad i, j \neq s.
\end{aligned} \tag{4.8}$$

For the special case of $s = 1$, the above commutation relations (4.6)-(4.8) recover those in [21, 23].

In order to apply the algebraic Bethe Ansatz method, in addition to the relevant commutation relations (4.6)-(4.8), one needs to construct a reference state associated with each s , which is the common eigenstate of the operators $\mathcal{A}^{(s)}$, $\mathcal{D}^{(s)}_i$ and is annihilated by the operators $\mathcal{C}_i^{(s)}$. In contrast to the trigonometric and rational cases with *diagonal* $K^\pm(u)$ [6], the usual highest-weight state

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix},$$

is no longer the pseudo-vacuum state. However, after the face-vertex transformations (3.18) and (3.19), the face type K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ *simultaneously* become diagonal. This suggests that one can translate the \mathbb{Z}_n Belavin model with non-diagonal K-matrices into the corresponding SOS model with *diagonal* K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ given by (3.18)-(3.19). Then one can construct the pseudo-vacuum in the “face language” and use the algebraic Bethe Ansatz method to diagonalize the transfer matrix.

One of the reference states corresponding to the case of $s = 1$ (or the first one of the n reference states (4.9) below) was found in [21] and yields the first set of eigenvalues of the transfer matrix. Here, we give the complete n reference states $\{|\Omega^{(s)}(\lambda)\rangle | s = 1, \dots, n\}$. For

each s , we propose⁴

$$|\Omega^{(s)}(\lambda)\rangle = \phi_{\lambda-(N-1)w\hat{s}, \lambda-Nw\hat{s}}(-z_1) \otimes \phi_{\lambda-(N-2)w\hat{s}, \lambda-(N-1)w\hat{s}}(-z_2) \cdots \otimes \phi_{\lambda, \lambda-w\hat{s}}(-z_N). \quad (4.9)$$

These states ($s = 1, \dots, n$) depend on the boundary parameters $\{\lambda_i\}$ and the inhomogeneous parameters $\{z_j\}$, but not on the boundary parameters ξ and $\bar{\xi}$. We find that the states given by (4.9) are *exactly* the reference states in the following sense,

$$\mathcal{A}^{(s)}(\lambda - Nw\hat{s}|u)|\Omega^{(s)}(\lambda)\rangle = k(\lambda|u; \xi)_s |\Omega^{(s)}(\lambda)\rangle, \quad (4.10)$$

$$\begin{aligned} \mathcal{D}^{(s)}_j(\lambda - Nw\hat{s}|u)|\Omega^{(s)}(\lambda)\rangle &= \delta_j^i f^{(s)}(u) k(\lambda|u + \frac{w}{2}; \xi - \frac{w}{2})_j \\ &\times \left\{ \prod_{k=1}^N \frac{\sigma(u + z_k)\sigma(u - z_k)}{\sigma(u + z_k + w)\sigma(u - z_k + w)} \right\} |\Omega^{(s)}(\lambda)\rangle, \\ &\quad i, j \neq s \end{aligned} \quad (4.11)$$

$$\mathcal{C}_i^{(s)}(\lambda - Nw\hat{s}|u)|\Omega^{(s)}(\lambda)\rangle = 0, \quad i \neq s \quad (4.12)$$

$$\mathcal{B}_i^{(s)}(\lambda - Nw\hat{s}|u)|\Omega^{(s)}(\lambda)\rangle \neq 0, \quad i \neq s. \quad (4.13)$$

Here $f^{(s)}(u)$ is given by

$$f^{(s)}(u) = \frac{\sigma(2u)\sigma(\lambda_s + u + w + \xi)}{\sigma(2u + w)\sigma(\lambda_s + u + \xi)}. \quad (4.14)$$

In order to apply the algebraic Bethe Ansatz method to diagonalize the transfer matrix, we need to assume $N = n \times l$ with l being a positive integer [21]. For convenience, let us introduce a set of integers:

$$N_i = (n - i) \times l, \quad i = 0, 1, \dots, n - 1, \quad (4.15)$$

and $\frac{n(n-1)}{2}l$ complex parameters $\{v_k^{(i)} | k = 1, 2, \dots, N_{i+1}, i = 0, 1, \dots, n-2\}$. As in the usual nested Bethe Ansatz method, the parameters $\{v_k^{(i)}\}$ will be used to specify the eigenvectors of the corresponding reduced transfer matrices. They will be constrained later by Bethe Ansatz equations. For convenience, we adopt the following convention:

$$v_k = v_k^{(0)}, \quad k = 1, 2, \dots, N_1. \quad (4.16)$$

⁴Such states played an important role in constructing extra centers of the elliptic algebra at roots of unity [34].

We will seek the common eigenvectors (i.e. the so-called Bethe states) of the transfer matrix by acting the creation operators $\mathcal{B}_i^{(s)}$ on the reference state $|\Omega^{(s)}(\lambda)\rangle$

$$|v_1, \dots, v_{N_1}\rangle^{(s)} = \sum_{i_1, \dots, i_{N_1} \neq s} F^{i_1, i_2, \dots, i_{N_1}} \mathcal{B}_{i_1}^{(s)}(\lambda + w(\hat{i}_1 - \hat{s})|v_1) \mathcal{B}_{i_2}^{(s)}(\lambda + w(\hat{i}_1 + \hat{i}_2 - 2\hat{s})|v_2) \\ \times \dots \mathcal{B}_{i_{N_1}}^{(s)}(\lambda + w \sum_{k=1}^{N_1} \hat{i}_k - wN_1\hat{s}|v_{N_1}) |\Omega^{(s)}(\lambda)\rangle. \quad (4.17)$$

The indices in the above equation should satisfy the following condition: the number of $i_k = j$, denoted by $\#(j)$, is l , i.e.

$$\#(j) = l, \quad j \neq s. \quad (4.18)$$

Then (3.1) and the above restriction (4.18) imply

$$\lambda + w \sum_{k=1}^{N_1} \hat{i}_k - wN_1\hat{s} = \lambda + wl \sum_{j \neq s} \hat{j} - wN_1\hat{s} = \lambda - w(l + N_1)\hat{s} = \lambda - wN\hat{s}, \quad (4.19)$$

which is crucial for the diagonalization of the transfer matrix in the remaining part of the paper.

With the help of (4.1), (4.4) and (4.5) we rewrite the transfer matrix (2.15) in terms of the operators $\mathcal{A}^{(s)}$ and $\mathcal{D}^{(s)i}_i$

$$\begin{aligned} \tau(u) &= \sum_{\nu=1}^n \tilde{k}(\lambda|u)_\nu \mathcal{T}(\lambda|u)_\nu^\nu \\ &= \tilde{k}(\lambda|u)_s \mathcal{A}^{(s)}(\lambda|u) + \sum_{i \neq s} \tilde{k}(\lambda|u)_i \mathcal{T}(\lambda|u)_i^i \\ &= \tilde{k}(\lambda|u)_s \mathcal{A}^{(s)}(\lambda|u) + \sum_{i \neq s} \tilde{k}(\lambda|u)_i R(2u, \lambda + w\hat{s})_{si}^{is} \mathcal{A}^{(s)}(\lambda|u) \\ &\quad + \sum_{i \neq s} \tilde{k}(\lambda|u)_i \left(\mathcal{T}(\lambda|u)_i^i - R(2u, \lambda + w\hat{s})_{si}^{is} \mathcal{A}^{(s)}(\lambda|u) \right) \\ &= \sum_{i=1}^n \tilde{k}(\lambda|u)_i R(2u, \lambda + w\hat{s})_{si}^{is} \mathcal{A}^{(s)}(\lambda|u) \\ &\quad + \sum_{i \neq s} \tilde{k}^{(1)}(\lambda|u + \frac{w}{2})_i \frac{\sigma(\lambda_{is} - w)}{\sigma(\lambda_{is})} \left(\mathcal{T}(\lambda|u)_i^i - R(2u, \lambda + w\hat{s})_{si}^{is} \mathcal{A}^{(s)}(\lambda|u) \right) \\ &= \alpha_s^{(1)}(u) \mathcal{A}^{(s)}(\lambda|u) + \sum_{i \neq s} \tilde{k}^{(1)}(\lambda|u + \frac{w}{2})_i \mathcal{D}^{(s)i}_i(\lambda|u). \end{aligned} \quad (4.20)$$

Here we have used (4.5) and introduced the function $\alpha_s^{(1)}(u)$,

$$\alpha_s^{(1)}(u) = \sum_{i=1}^n \tilde{k}(\lambda|u)_i R(2u, \lambda + w\hat{s})_{si}^{is}, \quad (4.21)$$

and the reduced K-matrix $\tilde{\mathcal{K}}^{(1)}(\lambda|u)$ with elements given by

$$\tilde{\mathcal{K}}^{(1)}(\lambda|u)_i^j = \delta_i^j \tilde{k}^{(1)}(\lambda|u)_i, \quad i, j \neq s \quad (4.22)$$

$$\tilde{k}^{(1)}(\lambda|u)_i = \left\{ \prod_{k \neq i, s}^n \frac{\sigma(\lambda_{ik} - w)}{\sigma(\lambda_{ik})} \right\} \frac{\sigma(\lambda_i + \bar{\xi} + u + \frac{(n-1)w}{2})}{\sigma(\lambda_i + \bar{\xi} - u - \frac{(n-1)w}{2})}, \quad i \neq s. \quad (4.23)$$

Using the technique developed in [21], after tedious calculations, we find that with the coefficients $F^{i_1, i_2, \dots, i_{N_1}}$ in (4.17) properly chosen, the Bethe state $|v_1, \dots, v_{N_1}\rangle^{(s)}$ becomes the eigenstate of the transfer matrix (2.15),

$$\tau(u)|v_1, \dots, v_{N_1}\rangle^{(s)} = \Lambda_s(u; \xi, \{v_k\})|v_1, \dots, v_{N_1}\rangle^{(s)}, \quad (4.24)$$

provided that the parameters $\{v_k^{(i)} | k = 1, 2, \dots, N_{i+1}, i = 0, 1, \dots, n-2\}$ satisfy the following Bethe Ansatz equations:

$$\begin{aligned} \beta_s^{(1)}(v_j) \frac{\sigma(2v_j + w)}{\sigma(2v_j + 2w)} \prod_{k \neq j, k=1}^{N_1} \frac{\sigma(v_j + v_k)\sigma(v_j - v_k - w)}{\sigma(v_j + v_k + 2w)\sigma(v_j - v_k + w)} \\ = \prod_{k=1}^N \frac{\sigma(v_j + z_k)\sigma(v_j - z_k)}{\sigma(v_j + z_k + w)\sigma(v_j - z_k + w)} \Lambda_s^{(1)}(v_j + \frac{w}{2}; \xi - \frac{w}{2}, \{v_a^{(1)}\}), \end{aligned} \quad (4.25)$$

$$\begin{aligned} \beta_s^{(i+1)}(v_j^{(i)}) \frac{\sigma(2v_j^{(i)} + w)}{\sigma(2v_j^{(i)} + 2w)} \prod_{k \neq j, k=1}^{N_{i+1}} \frac{\sigma(v_j^{(i)} + v_k^{(i)})\sigma(v_j^{(i)} - v_k^{(i)} - w)}{\sigma(v_j^{(i)} + v_k^{(i)} + 2w)\sigma(v_j^{(i)} - v_k^{(i)} + w)} \\ = \prod_{k=1}^{N_i} \frac{\sigma(v_j^{(i)} + z_k^{(i)})\sigma(v_j^{(i)} - z_k^{(i)})}{\sigma(v_j^{(i)} + z_k^{(i)} + w)\sigma(v_j^{(i)} - z_k^{(i)} + w)} \Lambda_s^{(i+1)}(v_j^{(i)} + \frac{w}{2}; \xi^{(i)} - \frac{w}{2}, \{v_a^{(i+1)}\}), \\ i = 1, \dots, n-2. \end{aligned} \quad (4.26)$$

Here $\{\beta_s^{(i)}(u) | i = 1, \dots, n-1\}$ are functions given in Appendix A, $\{\Lambda_s^{(i)}(u; \xi, \{v_k^{(i)}\}) | i = 1, \dots, n-1\}$ are the eigenvalues (given in Appendix A) of the reduced transfer matrices in the nested Bethe Ansatz process, and the reduced boundary parameters $\{\xi^{(i)}\}$ and inhomogeneous parameters $\{z_k^{(i)}\}$ are given by

$$\xi^{(i+1)} = \xi^{(i)} - \frac{w}{2}, \quad z_k^{(i+1)} = v_k^{(i)} + \frac{w}{2}, \quad i = 0, \dots, n-2. \quad (4.27)$$

In the above we have adopted the convention: $\xi = \xi^{(0)}$, $z_k^{(0)} = z_k$. The corresponding eigenvalue $\Lambda_s(u; \xi, \{v_k\})$ is given by

$$\Lambda_s(u; \xi, \{v_k\}) = \beta_s^{(1)}(u) \frac{\sigma(\lambda_s + \xi + u + w)}{\sigma(\lambda_s + \xi + u)} \prod_{k=1}^{N_1} \frac{\sigma(u + v_k)\sigma(u - v_k - w)}{\sigma(u + v_k + w)\sigma(u - v_k)}$$

$$\begin{aligned}
& + \frac{\sigma(2u)\sigma(\lambda_s + u + w + \xi)}{\sigma(2u + w)\sigma(\lambda_s + u + \xi)} \prod_{k=1}^{N_1} \frac{\sigma(u - v_k + w)\sigma(u + v_k + 2w)}{\sigma(u - v_k)\sigma(u + v_k + w)} \\
& \times \prod_{k=1}^{N_0} \frac{\sigma(u + z_k)\sigma(u - z_k)}{\sigma(u + z_k + w)\sigma(u - z_k + w)} \Lambda_s^{(1)}\left(u + \frac{w}{2}; \xi - \frac{w}{2}, \{v_i^{(1)}\}\right). \quad (4.28)
\end{aligned}$$

It is easy to check that the first set of eigenvalues $\Lambda_1(u; \xi, \{v_k^{(i)}\})$ and the corresponding Bethe Ansatz equations are exactly those found in [21]. However, the rest $n - 1$ sets of eigenvalues $\{\Lambda_s(u; \xi, \{v_k\}) | s = 2, \dots, n - 1\}$ and the associated Bethe Ansatz equations are new ones. For the special case of $n = 2$, which corresponds to the open XYZ spin chain, we find that the two sets of eigenvalues $\{\Lambda_s(u; \xi, \{v_k\}) | s = 1, 2\}$, after rescaling of an overall factor due to different normalizations of the R- and K-matrices, recover those in [35] obtained by directly solving the T - Q relation (or functional Bethe Ansatz). As shown in [35], these two sets of eigenvalues give the complete spectrum of the transfer matrix of the open XYZ spin chain. As a consequence, the corresponding two sets of Bethe states $\{|v_1, \dots, v_{N_1}\rangle^{(s)} | s = 1, 2\}$ together constitute the complete eigenstates of the transfer matrix of the open XYZ model. Therefore, it is expected that the n sets of eigenvalues $\{\Lambda_s(u; \xi, \{v_k\}) | s = 1, \dots, n\}$ (4.28) [resp. Bethe states $\{|v_1, \dots, v_{N_1}\rangle^{(s)} | s = 1, \dots, n\}$ (4.17) and (4.25)-(4.26)] together give rise to the complete spectrum [resp. the complete eigenstates] of the transfer matrix (2.15) of the open \mathbb{Z}_n Belavin model.

5 Result for the associated Gaudin model

As will be seen from the definitions of the intertwiners (3.8), (3.10) and (3.11), specialized to $m = \lambda$, $\phi_{\lambda, \lambda - w\hat{i}}(u)$ and $\bar{\phi}_{\lambda, \lambda - w\hat{i}}(u)$ do not depend on w while $\tilde{\phi}_{\lambda, \lambda - w\hat{i}}(u)$ does. Consequently, the K-matrix $K^-(u)$ does not depend on the crossing parameter w , but $K^+(u)$ does. So we use the convention:

$$K(u) = \lim_{w \rightarrow 0} K^-(u) = K^-(u). \quad (5.1)$$

We further assume that the parameter $\bar{\xi}$ has the following behavior⁵ as $w \rightarrow 0$,

$$\bar{\xi} = \xi + w\delta + O(w^2), \quad (5.2)$$

⁵In [12, 23, 20], a special case of $\bar{\xi} = \xi$ was studied. The generalization to the case with nonvanishing δ is straightforward.

with a constant δ . It implies that

$$\lim_{w \rightarrow 0} \bar{\xi} = \xi.$$

Then the K-matrices satisfy the following relation

$$\lim_{w \rightarrow 0} \{K^+(u) K^-(u)\} = \lim_{w \rightarrow 0} \{K^+(u)\} K(u) = \text{id}. \quad (5.3)$$

Let us introduce the elliptic Gaudin operators $\{H_j | j = 1, 2, \dots, N\}$ associated with the inhomogeneous \mathbb{Z}_n Belavin model with open boundaries specified by the generic K-matrices (2.17) and (2.18):

$$H_j = \Gamma_j(z_j) + \sum_{k \neq j}^N r_{kj}(z_j - z_k) + K_j^{-1}(z_j) \left\{ \sum_{k \neq j}^N r_{jk}(z_j + z_k) \right\} K_j(z_j), \quad (5.4)$$

where $\Gamma_j(u) = \frac{\partial}{\partial w} \{\bar{K}_j(u)\}|_{w=0} K_j(u)$, $j = 1, \dots, N$, with $\bar{K}_j(u) = \text{tr}_0 \{K_0^+(u) R_{0j}^B(2u) P_{0j}\}$. Here $\{z_j\}$ are the inhomogeneous parameters of the inhomogeneous \mathbb{Z}_n Belavin model and $r(u)$ is given by (2.8). For a generic choice of the boundary parameters $\{\lambda_1, \dots, \lambda_n, \bar{\xi}\}$, $\Gamma_j(u)$ is a *non-diagonal* matrix.

Following [36, 37], the elliptic Gaudin operators (5.4) are obtained by expanding the double-row transfer matrix (2.15) at the point $u = z_j$ around $w = 0$:

$$\tau(z_j) = \tau(z_j)|_{w=0} + w H_j + O(w^2), \quad j = 1, \dots, N, \quad (5.5)$$

$$H_j = \frac{\partial}{\partial w} \tau(z_j)|_{w=0}. \quad (5.6)$$

The relations (2.7) and (5.3) imply that the first term $\tau(z_j)|_{w=0}$ in the expansion (5.5) is equal to an identity, namely,

$$\tau(z_j)|_{w=0} = \text{id}. \quad (5.7)$$

Then the commutativity of the transfer matrices $\{\tau(z_j)\}$ (2.16) for a generic w implies

$$[H_j, H_k] = 0, \quad i, j = 1, \dots, N. \quad (5.8)$$

Thus the elliptic Gaudin system defined by (5.4) is integrable. Moreover, the relation (5.6) between $\{H_j\}$ and $\{\tau(z_j)\}$ and the fact that the first term on the r.h.s. of (5.5) is identity operator enable us to extract the eigenstates of the elliptic Gaudin operators and the corresponding eigenvalues from the results obtained in the previous section.

Using (4.27), (A.3), (A.4) and (A.6)-(A.8), the Bethe Ansatz equations (4.25) and (4.26) become, respectively,

$$\begin{aligned}
& \beta_s^{(i+1)}(v_j^{(i)}) \frac{\sigma(2v_j^{(i)} + w)}{\sigma(2v_j^{(i)} + 2w)} \prod_{k \neq j, k=1}^{N_{i+1}} \frac{\sigma(v_j^{(i)} + v_k^{(i)})\sigma(v_j^{(i)} - v_k^{(i)} - w)}{\sigma(v_j^{(i)} + v_k^{(i)} + 2w)\sigma(v_j^{(i)} - v_k^{(i)} + w)} \\
&= \beta_s^{(i+2)}(v_j^{(i)} + \frac{w}{2}) \frac{\sigma(\lambda_{i+s+1} + \xi^{(i+1)} + v_j^{(i)} + \frac{3}{2}w)}{\sigma(\lambda_{i+s+1} + \xi^{(i+1)} + v_j^{(i)} + \frac{1}{2}w)} \\
&\quad \times \prod_{k=1}^{N_i} \frac{\sigma(v_j^{(i)} + v_k^{(i-1)} + \frac{w}{2})\sigma(v_j^{(i)} - v_k^{(i-1)} - \frac{w}{2})}{\sigma(v_j^{(i)} + v_k^{(i-1)} + \frac{3w}{2})\sigma(v_j^{(i)} - v_k^{(i-1)} + \frac{w}{2})} \\
&\quad \times \prod_{k=1}^{N_{i+2}} \frac{\sigma(v_j^{(i)} + v_k^{(i+1)} + \frac{w}{2})\sigma(v_j^{(i)} - v_k^{(i+1)} - \frac{w}{2})}{\sigma(v_j^{(i)} + v_k^{(i+1)} + \frac{3w}{2})\sigma(v_j^{(i)} - v_k^{(i+1)} + \frac{w}{2})}, \\
&\quad i = 0, \dots, n-3,
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
& \beta_s^{(n-1)}(v_j^{(n-2)}) \frac{\sigma(2v_j^{(n-2)} + w)}{\sigma(2v_j^{(n-2)} + 2w)} \prod_{k \neq j, k=1}^{N_{n-1}} \frac{\sigma(v_j^{(n-2)} + v_k^{(n-2)})\sigma(v_j^{(n-2)} - v_k^{(n-2)} - w)}{\sigma(v_j^{(n-2)} + v_k^{(n-2)} + 2w)\sigma(v_j^{(n-2)} - v_k^{(n-2)} + w)} \\
&= \frac{\sigma(\lambda_{s-1} + \bar{\xi} + v_j^{(n-2)} + w)\sigma(\lambda_{s-1} + \xi^{(n-1)} - v_j^{(n-2)} - \frac{w}{2})}{\sigma(\lambda_{s-1} + \bar{\xi} - v_j^{(n-2)} - w)\sigma(\lambda_{s-1} + \xi^{(n-1)} + v_j^{(n-2)} + \frac{w}{2})} \\
&\quad \times \prod_{k=1}^{N_{n-2}} \frac{\sigma(v_j^{(n-2)} + v_k^{(n-3)} + \frac{w}{2})\sigma(v_j^{(n-2)} - v_k^{(n-3)} - \frac{w}{2})}{\sigma(v_j^{(n-2)} + v_k^{(n-3)} + \frac{3w}{2})\sigma(v_j^{(n-2)} - v_k^{(n-3)} + \frac{w}{2})}.
\end{aligned} \tag{5.10}$$

Here we have used the convention: $v_k^{(-1)} = z_k$, $k = 1, \dots, N$. The quasi-classical property (3.7) of $R(u, m)$, (A.1) and (A.2) lead to the following relations

$$\beta_s^{(i+1)}(u, \bar{\xi}, 0) = 1, \quad \frac{\partial}{\partial u} \beta_s^{(i+1)}(u, \bar{\xi}, 0) = 0, \quad i = 0, \dots, n-2. \tag{5.11}$$

Noticing the restriction (5.2), one may introduce some functions $\{\gamma_s^{(i+1)}(u)\}$ associated with $\{\beta_s^{(i+1)}(u, \bar{\xi}, w)\}$

$$\gamma_s^{(i+1)}(u) = \frac{\partial}{\partial w} \beta_s^{(i+1)}(u, \bar{\xi}, w)|_{w=0} + \delta \frac{\partial}{\partial \xi} \beta_s^{(i+1)}(u, \bar{\xi}, 0)|_{\bar{\xi}=\xi}, \quad i = 0, \dots, n-2. \tag{5.12}$$

Using (5.5), we can extract n sets of eigenvalues $\{h_j^{(s)} | s = 1, \dots, n\}$ (resp. the corresponding Bethe Ansatz equations) of the Gaudin operators H_j (5.4) from the expansion around $w = 0$ for the first order of w of the eigenvalues (4.28) of the transfer matrix $\tau(u = z_j)$ (resp. the Bethe Ansatz equations (5.9) and (5.10)). Finally, the eigenvalues of the \mathbb{Z}_n elliptic Gaudin operators are

$$h_j^{(s)} = \gamma_s^{(1)}(z_j) + \zeta(\lambda_s + \xi + z_j) - \sum_{k=1}^{N_1} \{\zeta(z_j + x_k) + \zeta(z_j - x_k)\}, \tag{5.13}$$

where ζ -function is defined in (2.3). The $\frac{n(n-1)l}{2}$ parameters $\{x_k^{(i)} | k = 1, 2, \dots, N_{i+1}, i = 0, 1, \dots, n-2\}$ (including x_k as $x_k = x_k^{(0)}, k = 1, \dots, N_1$) are determined by the following Bethe Ansatz equations

$$\begin{aligned}
& \gamma_s^{(i+1)}(x_j^{(i)}) - \zeta(2x_j^{(i)}) - 2 \sum_{k \neq j, k=1}^{N_{i+1}} \left\{ \zeta(x_j^{(i)} + x_k^{(i)}) + \zeta(x_j^{(i)} - x_k^{(i)}) \right\} \\
& = \gamma_s^{(i+2)}(x_j^{(i)}) - \sum_{k=1}^{N_i} \left\{ \zeta(x_j^{(i)} + x_k^{(i-1)}) + \zeta(x_j^{(i)} - x_k^{(i-1)}) \right\} \\
& \quad + \zeta(\lambda_{i+s+1} + \xi + x_j^{(i)}) - \sum_{k=1}^{N_{i+2}} \left\{ \zeta(x_j^{(i)} + x_k^{(i+1)}) + \zeta(x_j^{(i)} - x_k^{(i+1)}) \right\}, \\
& \quad i = 0, \dots, n-3,
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
& \gamma_s^{(n-1)}(x_j^{(n-2)}) - \zeta(2x_j^{(n-2)}) - 2 \sum_{k \neq j, k=1}^{N_{n-1}} \left\{ \zeta(x_j^{(n-2)} + x_k^{(n-2)}) + \zeta(x_j^{(n-2)} - x_k^{(n-2)}) \right\} \\
& = \left(\delta + \frac{n}{2} \right) \zeta(\lambda_{s-1} + \xi + x_j^{(n-2)}) + \left(\frac{2-n}{2} - \delta \right) \zeta(\lambda_{s-1} + \xi - x_j^{(n-2)}) \\
& \quad - \sum_{k=1}^{N_{n-2}} \left\{ \zeta(x_j^{(n-2)} + x_k^{(n-3)}) + \zeta(x_j^{(n-2)} - x_k^{(n-3)}) \right\}.
\end{aligned} \tag{5.15}$$

Here we have used the convention: $x_k^{(-1)} = z_k, k = 1, \dots, N$ in (5.14). Then the n sets of eigenvalues $\{h_j^{(s)} | s = 1, \dots, n\}$ given by (5.13)-(5.15) (c.f. [23]) together constitute the complete spectrum of the Gaudin operators H_j (5.4).

6 Conclusions

We have discovered the multiple reference state structure of the \mathbb{Z}_n Belavin model with boundaries specified by the non-diagonal K-matrices $K^\pm(u)$, (2.17) and (2.18). It is found that there exist n reference states $\{|\Omega^{(s)}(\lambda)\rangle | s = 1, \dots, n\}$ (4.9), which lead to n sets of Bethe states $|v_1, \dots, v_{N_1}\rangle^{(s)}$ (4.17). These Bethe states give rise to n sets of Bethe Ansatz equations (4.25)-(4.26) and eigenvalues (4.28), labelled by $s = 1, \dots, n$. The first set of them, which corresponds to the $s = 1$ case, gives the results found in [21]. It is expected that these n sets of eigenvalues $\{\Lambda_s(u; \xi, \{v_k\}) | s = 1, \dots, n\}$ (4.28) together give rise to the complete spectrum of the transfer matrix $\tau(u)$ (2.15) for the \mathbb{Z}_n Belavin model with generic boundaries. In the quasi-classical limit (i.e. $w \rightarrow 0$), the resulting n sets of eigenvalues $\{h_j^{(s)} | s = 1, \dots, n\}$ given by (5.13)-(5.15) together constitute the complete spectrum of the Gaudin operators H_j (5.4).

Taking the scaling limit [38] of our general results, for the special $n = 2$ case, we recover the results obtained in [20] for the open XXZ spin chain. It is believed that such structure of multiple reference states also exists for the open $A_{n-1}^{(1)}$ trigonometric vertex model studied in [24].

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Appendix A: Definitions of the α -, β -, Λ - functions

In this appendix, we give the definitions of the functions $\alpha_s^{(i)}(u)$, $\beta_s^{(i)}(u)$ and $\Lambda^{(i)}(u; \xi, \{v_k^{(i)}\})$, which appeared in the expressions of the eigenvalues and the Bethe Ansatz equations (4.25)-(4.28).

In order to carry out the nested Bethe Ansatz for the \mathbb{Z}_n Belavin model with the generic open boundary conditions, one needs to introduce a set of reduced K-matrices $\tilde{\mathcal{K}}^{(r)}(\lambda|u)$ which include the original one $\tilde{\mathcal{K}}(\lambda|u) = \tilde{\mathcal{K}}^{(0)}(\lambda|u)$:

$$\begin{aligned}\tilde{\mathcal{K}}^{(r)}(\lambda|u)_i^j &= \delta_i^j \tilde{k}^{(r)}(\lambda|u)_i, \quad i, j = r+1, \dots, n, \quad r = 0, \dots, n-1, \\ \tilde{k}^{(r)}(\lambda|u)_i &= \left\{ \prod_{k \neq i, k=r+1}^n \frac{\sigma(\lambda_{ik} - w)}{\sigma(\lambda_{ik})} \right\} \frac{\sigma(\lambda_i + \bar{\xi} + u + \frac{(n-r)w}{2})}{\sigma(\lambda_i + \bar{\xi} - u - \frac{(n-r)w}{2})}.\end{aligned}$$

Moreover we introduce a set of functions $\{\alpha^{(r)}(u) | r = 1, \dots, n-1\}$ related to the reduced K-matrices $\tilde{\mathcal{K}}^{(r)}(\lambda|u)$

$$\alpha^{(r)}(u) = \sum_{i=r}^n R(2u, \lambda + w\hat{r})_{ri}^{ir} \tilde{k}^{(r-1)}(\lambda|u)_i, \quad r = 1, \dots, n-1, \quad (\text{A.1})$$

and an associated set of functions $\{\beta^{(i)}(u, \bar{\xi}, w) | i = 1, \dots, n-1\}$

$$\beta^{(i+1)}(u, \bar{\xi}, w) \equiv \beta^{(i+1)}(u) = \alpha^{(i+1)}(u) \frac{\sigma(\lambda_{i+1} + \xi - u - \frac{i}{2}w)}{\sigma(\lambda_{i+1} + \xi + u + w - \frac{i}{2}w)}, \quad i = 0, \dots, n-2. \quad (\text{A.2})$$

In the process of carrying out the nested Bethe Ansatz [21], one needs to introduce a set of functions $\{\Lambda^{(i)}(u; \xi, \{v_k^{(i)}\}) | i = 0, \dots, n-1\}$ which correspond to the eigenvalues of the reduced transfer matrices. The functions $\{\Lambda^{(i)}(u; \xi, \{v_k^{(i)}\})\}$ are given by the following recurrence relations

$$\Lambda^{(i)}(u; \xi^{(i)}, \{v_k^{(i)}\}) = \beta^{(i+1)}(u) \frac{\sigma(\lambda_{i+1} + \xi^{(i)} + u + w)}{\sigma(\lambda_{i+1} + \xi^{(i)} + u)} \prod_{k=1}^{N_{i+1}} \frac{\sigma(u + v_k^{(i)})\sigma(u - v_k^{(i)} - w)}{\sigma(u + v_k^{(i)} + w)\sigma(u - v_k^{(i)})}$$

$$\begin{aligned}
& + \frac{\sigma(2u)\sigma(\lambda_{i+1}+u+w+\xi^{(i)})}{\sigma(2u+w)\sigma(\lambda_{i+1}+u+\xi^{(i)})} \prod_{k=1}^{N_{i+1}} \frac{\sigma(u-v_k^{(i)}+w)\sigma(u+v_k^{(i)}+2w)}{\sigma(u-v_k^{(i)})\sigma(u+v_k^{(i)}+w)} \\
& \times \prod_{k=1}^{N_i} \frac{\sigma(u+z_k^{(i)})\sigma(u-z_k^{(i)})}{\sigma(u+z_k^{(i)}+w)\sigma(u-z_k^{(i)}+w)} \Lambda^{(i+1)}\left(u+\frac{w}{2}; \xi^{(i)}-\frac{w}{2}, \{v_j^{(i+1)}\}\right), \\
& \quad i = 1, \dots, n-2,
\end{aligned} \tag{A.3}$$

$$\Lambda^{(n-1)}(u; \xi^{(n-1)}) = \frac{\sigma(\lambda_n + \bar{\xi} + u + \frac{w}{2})\sigma(\lambda_n + \xi^{(n-1)} - u)}{\sigma(\lambda_n + \bar{\xi} - u - \frac{w}{2})\sigma(\lambda_n + \xi^{(n-1)} + u)}. \tag{A.4}$$

The reduced boundary parameters $\{\xi^{(i)}\}$ and inhomogeneous parameters $\{z_k^{(i)}\}$ are given by (4.27). It is remarked that all the functions $\{\alpha^{(i)}(u)\}$, $\{\beta^{(i)}(u)\}$ and $\{\Lambda^{(i)}(u; \xi, \{v_k^{(i)}\})\}$ are *indeed* the functions of the boundary parameters $\{\lambda_i\}$.

Since that the suffix index of the boundary parameters $\{\lambda_i\}$ takes value in \mathbb{Z}_n , one can introduce a \mathbb{Z}_n cyclic operator \mathcal{P} (i.e. $\mathcal{P}^n = \text{id}$), which acts on the space of functions of $\{\lambda_1, \dots, \lambda_n\}$. On any function $f(\lambda_1, \dots, \lambda_n)$ the action of the operator \mathcal{P} is given by

$$\mathcal{P}(f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n)) = f(\lambda_{1+1}, \dots, \lambda_n, \lambda_{n+1}) = f(\lambda_2, \dots, \lambda_n, \lambda_1). \tag{A.5}$$

Then we introduce the following α -, β -, Λ -functions:

$$\alpha_s^{(i)}(u) = \mathcal{P}^{s-1}(\alpha^{(i)}(u)), \quad s = 1, \dots, n, \quad i = 1, \dots, n-1, \tag{A.6}$$

$$\beta_s^{(i)}(u) = \mathcal{P}^{s-1}(\beta^{(i)}(u)), \quad s = 1, \dots, n, \quad i = 1, \dots, n-1, \tag{A.7}$$

$$\Lambda_s^{(i)}(u; \xi, \{v_k^{(i)}\}) = \mathcal{P}^{s-1} \Lambda^{(i)}(u; \xi, \{v_k^{(i)}\}), \quad s = 1, \dots, n, \quad i = 1, \dots, n-1. \tag{A.8}$$

Direct calculation shows that the two definitions of $\alpha_s^{(1)}$, (4.21) and (A.6), coincide with each other.

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